Exercise 1: A solid is subjected to the following motion:

$$x_1 = X_1, \qquad x_2 = X_2 + aX_3, \qquad x_3 = aX_2 + X_3$$

Determine the right Cauchy-Green deformation tensor ***C*** and the Green-Lagrange strain tensor ***E***.

Solution

We calculate first the corresponding deformation gradient tensor,

$$F_{ij} = \frac{\partial x_i}{\partial X_j} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \alpha \\ 0 & \alpha & 1 \end{pmatrix}$$

The deformation tensors of Cauchy-Green ***C*** and Green-Lagrange ***E*** are obtained using the definitions,

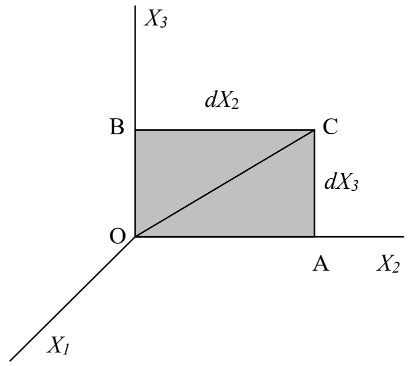
$$\mathbf{C} = \mathbf{F}^{\mathbf{T}}\mathbf{F}, \qquad \mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I})$$

In matrix form they are,

$$C_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \alpha \\ 0 & \alpha & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \alpha \\ 0 & \alpha & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+\alpha^2 & 2\alpha \\ 0 & 2\alpha & 1+\alpha^2 \end{pmatrix}$$

$$E_{ij} = \frac{1}{2}(C_{ij} - \delta_{ij}) = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha^2 & 2\alpha \\ 0 & 2\alpha & \alpha^2 \end{pmatrix}$$

Exercise 2: Consider the small rectangle in the following figure:



For the deformation given in Exercise 1,

1. Calculate the square of OA, OB and OC after deformation,
2. Calculate the variation of the squares of the segments OA, OB and OC,
3. Calculate the angle formed between segments OA and OB after deformation.

Solution

1. Using the symmetric tensor ***C*** obtained in exercise 1, the square of the length OC is

$$
\begin{aligned}
(ds)^2 &= C_{ij} dX_i dX_j \\
&= \begin{pmatrix} 0 & dX_2 & dX_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+a^2 & 2a \\ 0 & 2a & 1+a^2 \end{pmatrix} \begin{pmatrix} 0 \\ dX_2 \\ dX_3 \end{pmatrix} \\
&= (1+a^2)(dX_2)^2 + 4a dX_2 dX_3 + (1+a^2)(dX_3)^2
\end{aligned}
$$

Similarly, for OA et OB,

$$
(ds_2)^2 = \begin{pmatrix} 0 & dX_2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+\alpha^2 & 2\alpha \\ 0 & 2\alpha & 1+\alpha^2 \end{pmatrix} \begin{pmatrix} 0 \\ dX_2 \\ 0 \end{pmatrix} = (1+\alpha^2)(dX_2)^2
$$

$$
(ds_3)^2 = \begin{pmatrix} 0 & 0 & dX_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+\alpha^2 & 2\alpha \\ 0 & 2\alpha & 1+\alpha^2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ dX_3 \end{pmatrix} = (1+\alpha^2)(dX_3)^2
$$

2. The variations of the squares of the length of OA, OB et OC are,

$$\left(ds_2\right)^2 - \left(dX_2\right)^2 = a^2\left(dX_2\right)^2$$

$$\left(ds_3\right)^2 - \left(dX_3\right)^2 = a^2\left(dX_3\right)^2$$

$$\left(ds\right)^2 - (dX_2)^2 - \left(dX_3\right)^2 = a^2\left[\left(dX_2\right)^2 + \left(dX_3\right)^2\right] + 4adX_2dX_3$$

The same results are obtained using the expression,

$$\left(ds\right)^2 - \left(dS\right)^2 = 2E_{ij}dX_idX_j$$

3. The angle between OA and OB after deformation is

$$\cos\theta = \frac{\left(F\,dX_3e_3\right)\cdot\ \left(F\,dX_2e_2\right)}{\left(ds_3\right)\ \left(ds_2\right)} = \frac{\left(dX_3e_3\right)\cdot C\ \left(dX_2e_2\right)}{\left(ds_3\right)\ \left(ds_2\right)}$$

$$= \frac{\begin{pmatrix}0 & 0 & dX_3\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1+\alpha^2 & 2\alpha\\ 0 & 2\alpha & 1+\alpha^2\end{pmatrix}\begin{pmatrix}0\\ dX_2\\ 0\end{pmatrix}}{\left(1+\alpha^2\right)dX_2\,dX_3} = \frac{2\alpha}{1+\alpha^2}$$

Exercise 3: The deformation of a continuum is given by,

$$x_1 = X_1 + \gamma X_2 , \qquad x_2 = X_2 + \gamma X_3 , \qquad x_3 = X_3 + \gamma X_1$$

1. Calculate the Green-Lagrange ***E***, and the Euler-Almansi ***e***, deformation tensors.
2. Compare the two in the case where γ is very small.

Solution

The tensor **F** has the following components,

$$F_{ij} = \frac{\partial x_i}{\partial X_j} = \begin{pmatrix} 1 & \gamma & 0 \\ 0 & 1 & \gamma \\ \gamma & 0 & 1 \end{pmatrix}$$

Thus, ***C*** is given by,

$$C_{ij} = F_{mi} F_{mj} = \begin{pmatrix} 1 & 0 & \gamma \\ \gamma & 1 & 0 \\ 0 & \gamma & 1 \end{pmatrix} \begin{pmatrix} 1 & \gamma & 0 \\ 0 & 1 & \gamma \\ \gamma & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1+\gamma^2 & \gamma & \gamma \\ \gamma & 1+\gamma^2 & \gamma \\ \gamma & \gamma & 1+\gamma^2 \end{pmatrix}$$

For the tensor ***E*** we have,

$$E_{ij} = \frac{1}{2}\left(C_{ij} - \delta_{ij}\right) = \frac{1}{2} \begin{pmatrix} \gamma^2 & \gamma & \gamma \\ \gamma & \gamma^2 & \gamma \\ \gamma & \gamma & \gamma^2 \end{pmatrix}$$

To calculate the tensor of Euler-Almansi ***e***, we invert the F_{ij} :

$$F_{ij}^{-1} = \frac{1}{1+\gamma^3} \begin{pmatrix} 1 & -\gamma & \gamma^2 \\ \gamma^2 & 1 & -\gamma \\ -\gamma & \gamma^2 & 1 \end{pmatrix},$$

to obtain ***c*** as,

$$c_{ij}^{-1} = F_{mi}{}^{-1}F_{mj}{}^{-1} = \frac{1}{\left(1+\gamma^3\right)^2}\begin{pmatrix} 1 & \gamma^2 & -\gamma \\ -\gamma & 1 & \gamma^2 \\ \gamma^2 & -\gamma & 1 \end{pmatrix}\begin{pmatrix} 1 & -\gamma & \gamma^2 \\ \gamma^2 & 1 & -\gamma \\ -\gamma & \gamma^2 & 1 \end{pmatrix}$$

$$= \frac{1}{\left(1+\gamma^3\right)^2}\begin{pmatrix} 1+\gamma^2+\gamma^4 & -\gamma+\gamma^2-\gamma^3 & -\gamma+\gamma^2-\gamma^3 \\ -\gamma+\gamma^2-\gamma^3 & 1+\gamma^2+\gamma^4 & -\gamma+\gamma^2-\gamma^3 \\ -\gamma+\gamma^2-\gamma^3 & -\gamma+\gamma^2-\gamma^3 & 1+\gamma^2+\gamma^4 \end{pmatrix}.$$

Thus,

$$e_{ij} = \frac{1}{2}\left(\delta_{ij} - c_{ij}\right) = \frac{1}{2\left(1+\gamma^3\right)^2}\begin{pmatrix} -\gamma^2+2\gamma^3-\gamma^4+\gamma^6 & \gamma-\gamma^2+\gamma^3 & \gamma-\gamma^2+\gamma^3 \\ \gamma-\gamma^2+\gamma^3 & -\gamma^2+2\gamma^3-\gamma^4+\gamma^6 & \gamma-\gamma^2+\gamma^3 \\ \gamma-\gamma^2+\gamma^3 & \gamma-\gamma^2+\gamma^3 & -\gamma^2+2\gamma^3-\gamma^4+\gamma^6 \end{pmatrix}$$

when γ is very small, γ^2 and higher order terms are neglected and thus,

$$E_{ij} \approx e_{ij} \approx \frac{1}{2}\begin{pmatrix} 0 & \gamma & \gamma \\ \gamma & 0 & \gamma \\ \gamma & \gamma & 0 \end{pmatrix}.$$

Exercise 4: A displacement field dis given by,

$$u_1 = 3x_1x_2^2 \ , \qquad u_2 = 2x_3x_1 \ , \qquad u_3 = x_3^2 - x_1x_2$$

1. Determine the components of the infinitésimal strain tensor.
2. Verify the compatibility equations.

Solution

(a) Applying the definition, we obtain for ε_{ij},

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1} = 3x_2^2 \ , \qquad \varepsilon_{22} = \frac{\partial u_2}{\partial x_2} = 0 \ , \qquad \varepsilon_{33} = \frac{\partial u_3}{\partial x_3} = 2x_3 \ ,$$

$$\varepsilon_{12} = \varepsilon_{21} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = 3x_1x_2 + x_3 \ ,$$

$$\varepsilon_{13} = \varepsilon_{31} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) = -\frac{1}{2}x_2 \ ,$$

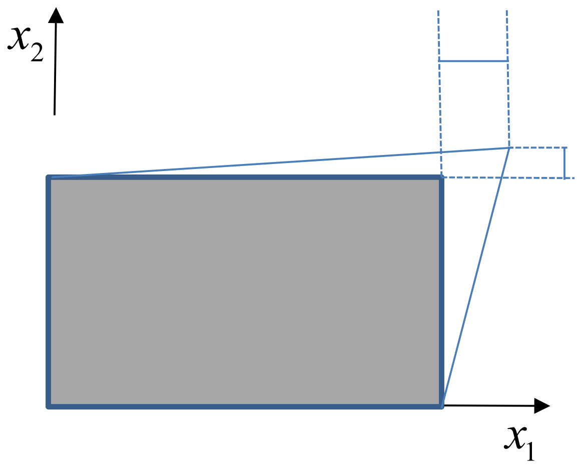
$$\varepsilon_{23} = \varepsilon_{32} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) = \frac{1}{2}x_1 \ ,$$

In matrix form,

$$\varepsilon_{ij} = \begin{pmatrix} 3x_2^2 & 3x_1x_2 + x_3 & -x_2/2 \\ 3x_1x_2 + x_3 & 0 & x_1/2 \\ -x_2/2 & x_1/2 & 2x_3 \end{pmatrix}$$

(b) Inserting these strains in B2.175, we see that the compatibility conditions are satisfied. This is expected since the strains are calculated from a given displacement field.

Exercise 5: A 3m by 2m thin plate is deformed by the movement of the corner point B to B' (see Figure). If the displacement field is given by $u_1 = c_1 x_1 x_2$, $u_2 = c_2 x_1 x_2$ where c_1, c_2 are constants, determine,



Dimensions: AB = CD = 3m;
AD = BC =2 m

1. The expressions for the displacements (identify c_1, c_2)
2. The strain components,
3. The normal strain along the direction BD

Solution (the displacements of point B are in mm)

1. Using the given coordinates of point B, we have,

$$u_1 = c_1 x_1 x_2 \Rightarrow 0.003m = c_1(3m)(2m) \Rightarrow c_1 = 500 \cdot 10^{-6} m^{-1}$$
$$u_2 = c_2 x_1 x_2 \Rightarrow 0.0015m = c_2(3m)(2m) \Rightarrow c_2 = 250 \cdot 10^{-6} m^{-1}$$

2. Using the definitions (B2.169), we obtain,

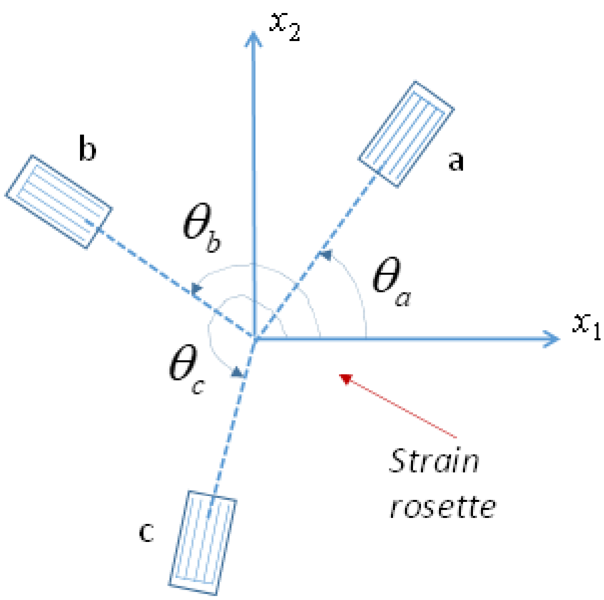
$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1} = 500 \cdot 10^{-6} x_2 = 500 x_2 \ \mu\varepsilon$$

$$\varepsilon_{22} = \frac{\partial u_2}{\partial x_2} = 250 \cdot 10^{-6} x_1 = 250 x_1 \ \mu\varepsilon$$

$$\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = \frac{1}{2}\left(500 \cdot 10^{-6} x_1 + 250 \cdot 10^{-6} x_2\right) = \frac{1}{2}\left(500 x_1 + 250 x_2\right)\mu\varepsilon$$

3. The angle in the diagonal direction is, $\theta = \tan^{-1}\frac{2}{3} = 33.7^o$. To obtain the strain along the diagonal direction, we use the expression (similar to the stress transformation),

$$\varepsilon_N = \varepsilon_{11}\cos^2\theta + \varepsilon_{22}\sin^2\theta + 2\varepsilon_{12}\cos\theta\sin\theta$$
$$= 500 \cdot 2 \cdot 0{,}832^2 + 250 \cdot 3 \cdot 0{,}555^2 + 2\frac{1}{2}(500 \cdot 3 + 250 \cdot 2)0{,}832 \cdot 0{,}555 = 1864.5\mu\varepsilon$$



Exercise 6: A 60° strain rosette reading system, shown in the figure below, is attached on the free surface a plate before loading. For this particular rosette,

$$\theta_a = 0^o \;;\; \theta_b = 60^o \;;\; \theta_c = 120^o$$

After loading, the strain readings are,

$$\varepsilon_a = 190\mu\varepsilon, \;\; \varepsilon_b = 200\mu\varepsilon, \;\; \varepsilon_c = -300\mu\varepsilon$$

Determine (a) the in plane principal strains and their directions, (b) the maximum shear strain.

Solution

The strains along each of these three directions are,

$$\varepsilon_a = \varepsilon_{11}\cos^2\theta_a + \varepsilon_{22}\sin^2\theta_a + 2\varepsilon_{12}\cos\theta_a\sin\theta_a$$
$$\varepsilon_b = \varepsilon_{11}\cos^2\theta_b + \varepsilon_{22}\sin^2\theta_b + 2\varepsilon_{12}\cos\theta_b\sin\theta_b$$
$$\varepsilon_c = \varepsilon_{11}\cos^2\theta_c + \varepsilon_{22}\sin^2\theta_c + 2\varepsilon_{12}\cos\theta_c\sin\theta_c$$

For the given rosette,

$$\theta_a = 0^o \;;\; \theta_b = 60^o \;;\; \theta_c = 120^o$$

we have the following strains,

$$\varepsilon_{11} = \varepsilon_a = 190 \;\mu\varepsilon$$
$$\varepsilon_{22} = \frac{1}{3}(2\varepsilon_b + 2\varepsilon_c - \varepsilon_a) = -130 \;\mu\varepsilon \quad \Rightarrow \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{12} & \varepsilon_{22} \end{pmatrix} = \begin{pmatrix} 190 & 288.5 \\ 288.5 & -130 \end{pmatrix}\mu\varepsilon$$
$$2\varepsilon_{12} = \frac{2}{\sqrt{3}}(\varepsilon_b - \varepsilon_c) = 577 \;\mu\varepsilon$$

With these strains, we can obtain the principal values using,

$$\varepsilon_{1,2} = \frac{\varepsilon_{11}+\varepsilon_{22}}{2} + \sqrt{\left(\frac{\varepsilon_{11}-\varepsilon_{22}}{2}\right)^2 + \varepsilon_{12}^2} \quad \Rightarrow \quad \varepsilon_1 = 360\mu\varepsilon, \quad \varepsilon_2 = -300\mu\varepsilon$$

Their orientation is,

$$2\theta_p = \tan^{-1}\frac{2\varepsilon_{12}}{\varepsilon_{11}-\varepsilon_{22}} \quad \Rightarrow \quad 2\theta_p = 61^o$$

The maximum shearing strain is, $\varepsilon_{12}^{\max} = \pm\frac{\varepsilon_1-\varepsilon_2}{2} = \pm 330\mu\varepsilon$.